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Oxford Cambridge and RSA

A Level Further Mathematics B (MEI)

Y420 Core Pure
Sample Question Paper

Model Answers.

Date – Morning/Afternoon
Time allowed: 2 hours 40 minutes

OCR supplied materials:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer all the questions.
- Write your answer to each question in the space provided in the Printed Answer Booklet.
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do not write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is 144.
- The marks for each question are shown in brackets [ ].
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of 24 pages. The Question Paper consists of 8 pages.

Section A (33 marks)

Answer all the questions.

1 Find the acute angle between the lines with vector equations $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. [3]

let $a = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

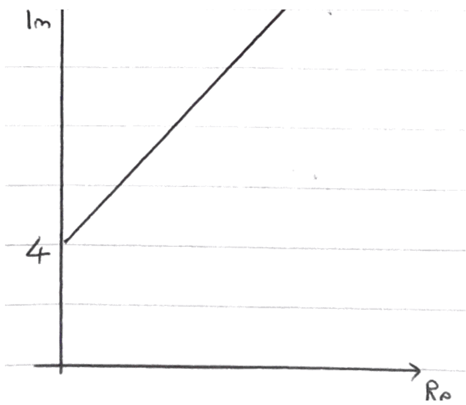
$a \cdot b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 3 + 2 + 2 = 7$

$|a| = \sqrt{1+4+1} = \sqrt{6}$ $|b| = \sqrt{9+1+4} = \sqrt{14}$

$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{7}{\sqrt{6}\sqrt{14}} = 0.7638..$

$\Rightarrow \theta = \underline{\underline{40.2^\circ}}$

2 (i) On an Argand diagram draw the locus of points which satisfy $\arg(z - 4i) = \frac{\pi}{4}$. [2]



(ii) Give, in complex form, the equation of the circle which has centre at $6 + 4i$ and touches the locus in part (i). [4]

let line from (i) be l

$l .$ $y = x + 4$

let circle is C

$C \cdot$ $(y-4)^2 + (x-6)^2 = r^2$

The line touches only once @:

$$(x+4-4)^2 + (x-6)^2 = r^2$$

$$x^2 + x^2 + 36 - 12x = r^2$$

$$\Rightarrow 2x^2 - 12x + 36 - r^2 = 0$$

$$b^2 - 4ac = 0$$

$$\Rightarrow 144 - 4(2)(36 - r^2) = 0$$

$$\Rightarrow 8r^2 = 144$$

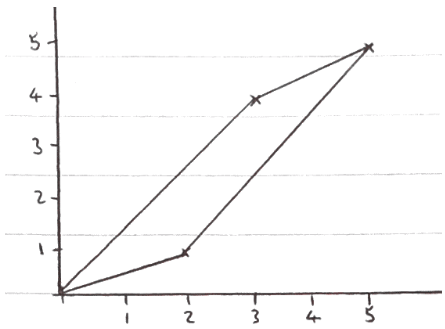
$$\Rightarrow r = \sqrt{18} = \underline{\underline{3\sqrt{2}}}$$

$\therefore$ Eq of Circle is: $(y-4)^2 + (x-6)^2 = 18$

$\Rightarrow |z - (6+4i)| = 3\sqrt{2}$ in complex form

3 Transformation M is represented by matrix $M = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

(i) On the diagram in the Printed Answer Booklet draw the image of the unit square under M . [2]



(ii) (A) Show that there is a constant k such that $M\begin{pmatrix} x \\ kx \end{pmatrix} = 5\begin{pmatrix} x \\ kx \end{pmatrix}$ for all x. [2]

$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = 5\begin{pmatrix} x \\ kx \end{pmatrix} \Rightarrow 2x + 3kx = 5x$ — (i)

and

$x + 4kx = 5kx$ — (ii)

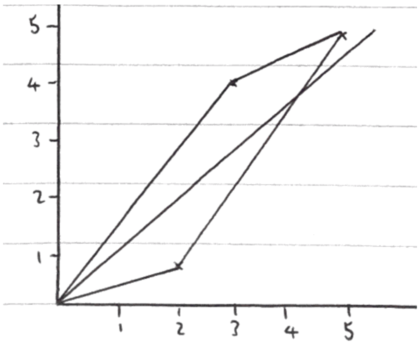
i) $3x = 3kx \Rightarrow k = 1$ and ii): $4kx = 4x \Rightarrow k = 1$

$\therefore \underline{\underline{k = 1}}$

(B) Hence find the equation of an invariant line under M . [1]

Invariant line is $\begin{pmatrix} x \\ x \end{pmatrix} \Rightarrow \underline{\underline{y = x}}$ is equation.

(C) Draw the invariant line from part (ii) (B) on your diagram for part (i). [1]



4 You are given that $z = 1 + 2i$ is a root of the equation $z^3 - 5z^2 + qz - 15 = 0$, where $q \in$.

Find

- the other roots,
- the value of q. [5]

$z = 1+2i$ is a root, $\therefore$ $z^* = 1-2i$ is also a root.

$\Sigma\alpha = -\frac{b}{a} \Rightarrow 1+2i + 1-2i + \alpha = -\frac{(-5)}{1}$

$\Rightarrow 2+\alpha = 5 \Rightarrow \underline{\underline{\alpha = 3}}$

$\Sigma\alpha\beta = c/a \Rightarrow (1+2i)(1-2i) + (1+2i)5 + 5(1-2i) = \frac{q}{1}$

$\Rightarrow 1 - 4i^2 + 3 + 6i + 3 - 6i = q$

$\Rightarrow \underline{\underline{11 = q}}$

$\therefore$ Other $\underline{\underline{\text{root} = 3}}$ and $\underline{\underline{q = 11}}$

5 (i) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions. [2]

$\dfrac{A}{r+1} + \dfrac{B}{r+3} \Rightarrow 2 = A(r+3) + B(r+1)$

Let $r = -3 \Rightarrow 2 = 0 + (-2B) \Rightarrow B = -1$

Let $r = -1 \Rightarrow 2 = 2A + 0 \Rightarrow A = 1$

$$\therefore \frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$$

(ii) Hence find $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$, expressing your answer as a single fraction. [5]

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+3}$$

$$= \frac{1}{2}\left(\frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+2} + \cancel{\frac{1}{n+1}} - \frac{1}{n+3} \right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$= \frac{1}{2}\left(\frac{3(n+2)(n+3) + 2(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)} \right)$$

$$= \frac{3n^2 + 15n + 18 + 2n^2 + 10n + 12 - 6n - 18 - 6n - 12}{12(n+2)(n+3)}$$

$$= \frac{5n^2 + 13n}{12(n+2)(n+3)}$$

6 (i) A curve is in the first quadrant. It has parametric equations $x = \cosh t + \sinh t$, $y = \cosh t - \sinh t$ where $t \in \mathbb{R}$. Show that the cartesian equation of the curve is $xy = 1$. [2]

$$x = \cosh t + \sinh t \qquad y = \cosh t - \sinh t$$

$$xy = (\cosh t + \sinh t)(\cosh t - \sinh t)$$

$$= \cosh^2 t - \sinh^2 t$$

$$= 1 \Rightarrow \underline{\underline{xy = 1}}$$

Fig. 6 shows the curve from part (i). P is a point on the curve. O is the origin. Point A lies on the x-axis, point B lies on the y-axis and OAPB is a rectangle.

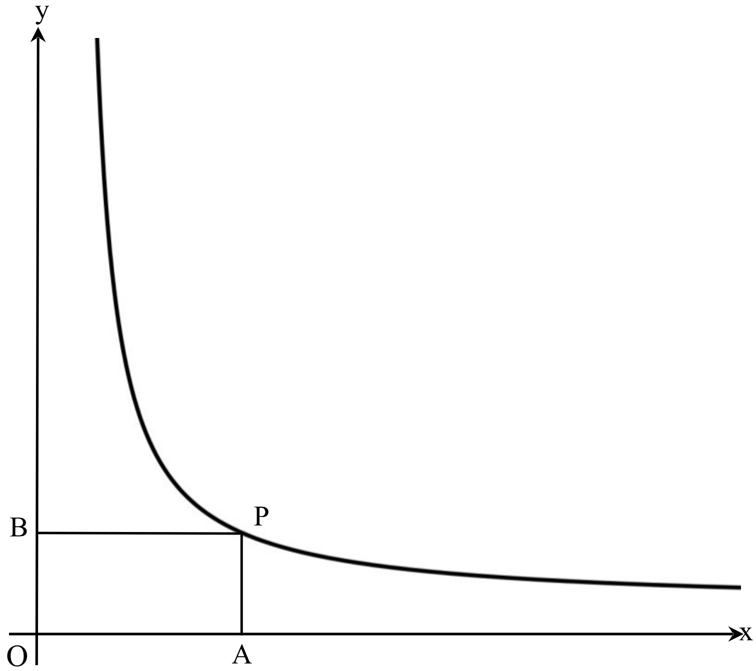


Fig. 6

(ii) Find the smallest possible value of the perimeter of rectangle OAPB. Justify your answer. [4]

Perimeter $= 2(x+y)$

$= 2(\cosh t + \cancel{\sinh t} + \cosh t - \cancel{\sinh t})$

$= 4\cosh t$

Minimum value of $\cosh t = 1$

$\Rightarrow$ Minimum value of perimeter $= \underline{\underline{4 \text{ units}}}$.

Section B (111 marks)

Answer all the questions

7 (i) Use the Maclaurin series for $\ln(1+x)$ up to the term in x^3 to obtain an approximation to $\ln 1.5$. [2]

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\ln(1+0.5) = 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \ldots$$

$$\Rightarrow \ln(1.5) = \sim 0.4167$$

(ii) (A) Find the error in the approximation in part (i). [1]

$$0.4167 - \ln 1.5 = 0.0112$$

(B) Explain why the Maclaurin series in part (i), with $x = 2$, should not be used to find an approximation to ln 3. [1]

The series is only valid for $-1 < x \leq 1$. To find $\ln 3$, x must $= 2$ which is outside of this range.

(iii) Find a cubic approximation to $\ln\left(\frac{1+x}{1-x}\right)$. [2]

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$= 2x + \frac{2x^3}{3}$$

(iv) (A) Use the approximation in part (iii) to find approximations to

- ln 1.5 and
- ln 3. [3]

$$\frac{1+x}{1-x} = 1.5 \Rightarrow 1 + x = 1.5 - 1.5x$$

$$\Rightarrow 2.5x = 0.5 \Rightarrow x = 0.2$$

$$\ln(1.5) = \ln\left(\frac{1+0.2}{1-0.2}\right) = 0.4 + \frac{2(0.2)^3}{3}$$

$$= 0.4053$$

$$\frac{1+x}{1-x} = 3 \Rightarrow 1 + x = 3 - 3x \Rightarrow x = 0.5$$

$$\ln(3) = \ln\left(\frac{1+0.5}{1-0.5}\right) = 1 + \frac{2(0.5)^3}{3} = 1.083$$

(B) Comment on your answers to part (iv) (A). [2]

Error for $\ln 1.5$: $\ln 1.5 - 0.4053 = 0.0001318$.
This is smaller than before. ∴ It is a better estimate. For $\ln 3$, x value was in the radius of convergence, so approximation is valid.

8 Find the cartesian equation of the plane which contains the three points $(1, 0, -1)$, $(2, 2, 1)$ and $(1, 1, 2)$. [5]

Let $A = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

$\overrightarrow{AB}$ & $\overrightarrow{AC}$ are $\parallel$ to the plane, so $AB \times AC$ will be normal to the plane.

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-2 \\ 0-3 \\ 1-0 \end{pmatrix} \Rightarrow n = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

Equation is: $4x - 3y + z = d$

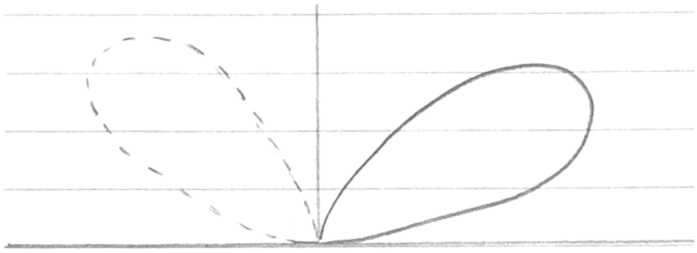
Subs, $(1, 0, -1)$ $\Rightarrow$ $4 + 0 - 1 = d$

$\Rightarrow d = 3$

$\Rightarrow$ $\underline{\underline{4x - 3y + z = 3}}$

9 A curve has polar equation $r = a\sin 3\theta$ for $-\frac{1}{3}\pi \leq \theta \leq \frac{1}{3}\pi$, where a is a positive constant.

(i) Sketch the curve. [2]



(ii) In this question you must show detailed reasoning.

Find, in terms of a and π, the area enclosed by one of the loops of the curve. [5]

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} a^2\sin^2 3\theta \ d\theta$$

$$\cos 6\theta = 1 - 2\sin^2 3\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}a^2 - \frac{1}{2}a^2\cos 6\theta \ d\theta$$

$$\Rightarrow \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$$

$$\Rightarrow \frac{a^2}{4}\left(\frac{\pi}{3} - \frac{1}{6}\times\sin(2\pi)\right) - 0$$

$$\Rightarrow \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right)$$

$$\Rightarrow \frac{\pi a^2}{12}$$

10 (i) Obtain the solution to the differential equation

$$x\frac{dy}{dx}+3y=\frac{1}{x}\text{ , where } x>0 \text{ ,}$$

given that $y=1$ when $x=1$. [7]

$$x\frac{dy}{dx}+3y=\frac{1}{x}$$

Div. by x:

$$\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^2}$$

$$\text{If} \Rightarrow e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

Mult both sides by x^3

$$\Rightarrow \frac{dy}{dx}x^3 + 3x^2y = x$$

$$\int \frac{d}{dx}(x^3y) = \int x \, dx$$

$$\Rightarrow x^3y = \frac{x^2}{2}+c$$

when $x=1$ $y=1$

$$\Rightarrow 1 = \frac{1}{2}+c \Rightarrow c = 1/2$$

$$\Rightarrow x^3y = \frac{x^2}{2}+1/2 \Rightarrow x^3y = \frac{x^2+1}{2}$$

$$\Rightarrow y = \frac{1+x^2}{2x^3}$$

(ii) Deduce that y decreases as x increases. [2]

$y = \frac{1}{2x^3} + \frac{x^2}{2x^3}$ as $x\uparrow$ both fractions $\downarrow$, $\therefore$ their sum, i.e y must also decrease.

11 (i) It is conjectured that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}=a-\frac{b}{n!},$$

where a and b are constants, and n is an integer such that $n \geq 2$.

By considering particular cases, show that if the conjecture is correct then $a = b = 1$. [2]

When $n=2$, $\frac{1}{2!} = a - \frac{b}{2!} \Rightarrow 1 = 2a - b$ — (i)

When $n=3$, $\frac{1}{2!} + \frac{2}{3!} = a - \frac{b}{3!} \Rightarrow 3+2 = 6a - b$

$\Rightarrow 5 = 6a - b$ — (ii)

Solving: $1-2a = 5-6a \Rightarrow \underline{\underline{a=1}}$

$\Rightarrow b = 2a-1 = 2-1 = 1$ $\therefore$ $\underline{\underline{a=b=1}}$

(ii) Use induction to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}=1-\frac{1}{n!} \text{ for } n \geq 2.$$ [7]

$n=2$, $\frac{1}{2!} = 1 - \frac{1}{2!} \Rightarrow \frac{1}{2} = \frac{1}{2}$

$\therefore$ Result is true for $n=2$

Assuming it's true for $n=k \Rightarrow$

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k-1}{k!} = 1 - \frac{1}{k!}$

Checking for $n = k+1$,

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k-1)!}$

$= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k-1}{k!}\right) + \frac{k}{(k+1)!}$

$= 1 - \frac{1}{k!} + \frac{k}{(k+1)!}$

$$= 1 + \frac{-(k+1)+k}{(k+1)!} = 1 - \frac{1}{(k+1)!} \quad \text{shown!}$$

$\therefore$ The result is true for $n = k+1$ if it is true for $n = k$. Because it's true for $n = 2$, it must be true for positive integers greater than or equal to 2, by mathematical induction.

12 In this question you must show detailed reasoning.

(i) Given that $y = \arctan x$, show that $\frac{dy}{dx} = \frac{1}{1+x^2}$. [3]

$y = \arctan x$

$\Rightarrow x = \tan y$

Differentiating Implicitly:

$1 = \frac{dy}{dx} \sec^2 x$

$\frac{1}{\sec^2 x} = \frac{dy}{dx}$

$\Rightarrow \frac{1}{1+\tan^2 x} = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ shown.

Fig. 12 shows the curve $y = \frac{1}{1+x^2}$.

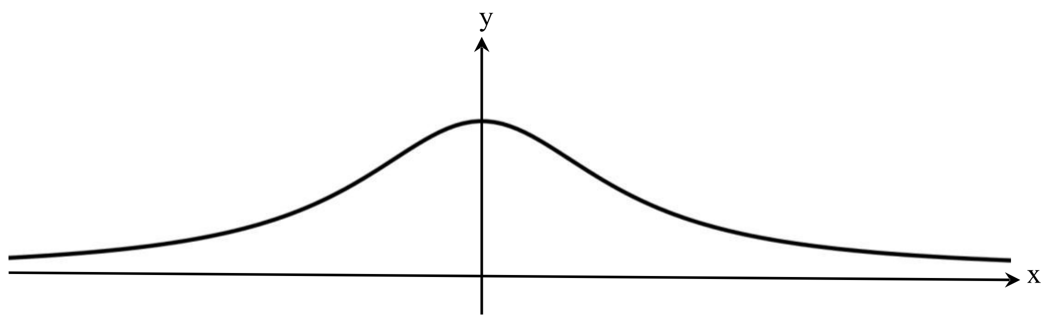


Fig. 12

(ii) Find, in exact form, the mean value of the function $f(x) = \frac{1}{1+x^2}$ for $-1 \le x \le 1$. [3]

$$\frac{1}{1-(-1)} \left[\arctan x \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \pi/4$$

(iii) The region bounded by the curve, the x-axis, and the lines $x = 1$ and $x = -1$ is rotated through 2π radians about the x-axis. Find, in exact form, the volume of the solid of revolution generated. [7]

$$\text{Vol} = \pi \int_{-1}^{1} \frac{1}{(1+x^2)^2}\, dx$$

let $x = \tan u$ $\Rightarrow dx = \sec^2 u \, du$

$\frac{dx}{du} = \sec^2 u$

Limits: $x = 1$ $u = \pi/4$

$x = -1$ $u = -\pi/4$

$$\Rightarrow \pi \int_{-\pi/4}^{\pi/4} \frac{1}{(1+\tan^2 u)^2} \times \sec^2 u \, du$$

$$\Rightarrow \pi \int_{-\pi/4}^{\pi/4} \frac{1}{\sec^4 u} \times \sec^2 u \, du$$

$$\Rightarrow \pi \int_{-\pi/4}^{\pi/4} \frac{1}{\sec^2 u} \, du$$

$$\Rightarrow \pi \int_{-\pi/4}^{\pi/4} \cos^2 u \, du = \pi \int_{-\pi/4}^{\pi/4} \frac{1}{2} + \frac{1}{2}\cos 2u \, du$$

$$\Rightarrow \frac{\pi}{2}\left[u + \frac{1}{2}\sin 2u \right]_{-\pi/4}^{\pi/4}$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{4} - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)\right]$$

$$\Rightarrow \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{4} + \frac{1}{2} \right]$$

$$\Rightarrow \frac{\pi}{4}(\pi + 2)$$

13 Matrix M is given by $M = \begin{pmatrix} k & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix}$, where k is a constant.

(i) Show that $\det M = 12(k-3)$. [2]

$$\det M = k(6+6) - (4-3) - 5(4+3)$$
$$= 12k - 1 - 35$$
$$= 12(k-3)$$

(ii) Find a solution of the following simultaneous equations for which $x \neq z$.

$$4x^2 + y^2 - 5z^2 = 6$$
$$2x^2 + 3y^2 - 3z^2 = 6$$
$$-x^2 + 2y^2 + 2z^2 = -6$$

[3]

In this case, $k = 4$, $\therefore \det M = 12$

$$\therefore \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & -3 \\ -1 & 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1/12 & 1/4 & 1/6 \\ 7/12 & -3/4 & 5/6 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 6 - 6 - 6 \\ -1/2 + 6/4 - 1 \\ 7/2 - 9/2 - 5 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow \begin{aligned} x &= \sqrt{6}\, i \\ y &= 0 \\ z &= \sqrt{6}\, i \end{aligned}$$

(iii) (A) Verify that the point (2, 0, 1) lies on each of the following three planes.

$$3x + y - 5z = 1$$
$$2x + 3y - 3z = 1$$
$$-x + 2y + 2z = 0$$

[1]

$3(2) + 0 - 5 = 1$ ✓
$2(2) + 0 - 3(1) = 1$ ✓
$-2 + 0 + 2(1) = 0$ ✓

(B) Describe how the three planes in part (iii) (A) are arranged in 3-D space. Give reasons for your answer. [4]

$k = 3$, the equations form matrix.

$\det M = 12(3-3) = 0$ ∴ Matrix is singular, i.e. no unique solution.

The planes are distinct, ∴ form a sheaf.

(iv) Find the values of k for which the transformation represented by M has a volume scale factor of 6. [3]

$\det M = 6$
$12(k-3) = 6$
$k - 3 = 0.5$
$k = \underline{\underline{3.5}}$

$\det M = -6$
$12(k-3) = -6$
$k - 3 = -0.5$
$k = \underline{\underline{2.5}}$

14 (i) Starting with the result

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

show that

(A) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ [2]

$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{in\theta} = (\cos\theta + i\sin\theta)^n$

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ shown.

(B) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. [2]

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$
$= \cos\theta - i\sin\theta$

$e^{i\theta} - e^{-i\theta} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$
$= 2\cos\theta$
$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ $\underline{\text{shown}}$.

(ii) Using the result in part (i) (A), obtain the values of the constants a, b, c and d in the identity

$$\cos 6\theta \equiv a\cos^6\theta + b\cos^4\theta + c\cos^2\theta + d\,. \qquad [6]$$

$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i\sin\theta)^2 + 20\cos^3\theta(i\sin\theta)^3$
$+ 15\cos^2\theta(i\sin\theta)^4 + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$
$+ 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$

Equating real part.

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$
$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2$
$-(1-\cos^2\theta)(1-\cos^2\theta)^2$
$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta$
$-(1 - 2\cos^2\theta + \cos^4\theta - \cos^2\theta + 2\cos^4\theta - \cos^6\theta)$
$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

(iii) Using the result in part (i) (B), obtain the values of the constants P, Q, R and S in the identity

$$\cos^6\theta \equiv P\cos 6\theta + Q\cos 4\theta + R\cos 2\theta + S\,. \qquad [5]$$

$\cos^6\theta = \left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right)^6 \qquad \text{let } z = e^{i\theta}$

$\Rightarrow \cos^6\theta = \frac{1}{2^6}\left(z + \frac{1}{z}\right)^6 = \frac{1}{64}\left(z^6 + \frac{6z^5}{z} + \frac{15z^4}{z^2} + \frac{20z^3}{z^3} + \frac{15z^2}{z^4}\right.$
$\left. + \frac{6z}{z^5} + \frac{1}{z^6}\right) = \frac{1}{64}\left(z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right)$

$= \frac{1}{64}\left(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\right)$

$= \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

(iv) Show that $\cos\frac{\pi}{12} = \left(\frac{26+15\sqrt{3}}{64}\right)^{\frac{1}{6}}$. [3]

$$\cos^6\frac{\pi}{12} = \frac{1}{32}\cos\frac{\pi}{2} + \frac{3}{16}\cos\frac{\pi}{3} + \frac{15}{32}\cos\frac{\pi}{6} + \frac{5}{16}$$

$$= 0 + \frac{3}{32} + \frac{15\sqrt{3}}{64} + \frac{5}{16}$$

$$= \frac{6+15\sqrt{3}+20}{64}$$

$$\cos^6\frac{\pi}{12} = \left(\frac{26+15\sqrt{3}}{64}\right)$$

$$\Rightarrow \cos\frac{\pi}{12} = \left(\frac{26+15\sqrt{3}}{64}\right)^{1/6}$$ shown.

15 In this question you must show detailed reasoning.

Show that

$$\int_0^{\frac{2}{3}} \operatorname{arsinh} 2x \, dx = \frac{2}{3}\ln 3 - \frac{1}{3}.$$ [8]

Using integration by parts.

$u = \operatorname{arsinh} 2x \qquad v' = 1$

$u' = \frac{2}{\sqrt{1+4x^2}} \qquad v = x$

$$\int_0^{2/3} \operatorname{arsinh} 2x = \left[x \operatorname{arsinh} 2x\right]_0^{2/3} - \int_0^{2/3} \frac{2x}{\sqrt{1+4x^2}}\,dx$$

$$= \frac{2}{3}\operatorname{arsinh}\frac{4}{3} - \left[\frac{1}{2}\left(1+4x^2\right)^{1/2}\right]^{2/3}$$

$$= \frac{2}{3}\operatorname{arsinh}\frac{4}{3} - \frac{1}{2}\sqrt{1+\frac{16}{9}} + \frac{1}{2} = \frac{2}{3}\ln\left(\frac{4}{3}+\sqrt{1+\frac{16}{9}}\right) - \frac{5}{6} + \frac{1}{2}$$

$$= \frac{2}{3}\ln\left(\frac{9}{3}\right) - \frac{5}{6} + \frac{1}{2} = \frac{2}{3}\ln 3 - 1/3$$ shown.

16 A small object is attached to a spring and performs oscillations in a vertical line. The displacement of the object at time t seconds is denoted by x cm.

Preliminary observations suggest that the object performs simple harmonic motion (SHM) with a period of 2 seconds about the point at which $x = 0$.

(i) (A) Write down a differential equation to model this motion. [3]

$T = \frac{2\pi}{\omega} \Rightarrow 2 = \frac{2\pi}{\omega} \Rightarrow \pi = \omega$

$\Rightarrow \frac{d^2x}{dt^2} = -\pi^2 x$ as $(a = -\omega^2 x)$

(B) Give the general solution of the differential equation in part (i) (A). [1]

$x = A\cos \pi t + B\sin \pi t$

Subsequent observations indicate that the object's motion would be better modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + (k^2 + 9)x = 0 \qquad (*)$$

where k is a positive constant.

(ii) (A) Obtain the general solution of (*). [3]

Auxiliary eqn. $\lambda^2 + 2K\lambda + K^2 + 9 = 0$

$\lambda = \frac{-2K \pm \sqrt{4K^2 - 4(K^2+9)}}{2}$

$\lambda = \frac{-2K \pm \sqrt{-36}}{2} = -K \pm 3i$

$\therefore x = e^{-Kt}(A\cos 3t + B\sin 3t)$

(B) State two ways in which the motion given by this model differs from that in part (i). [2]

Period is more than 2s and the motion is damped.

The amplitude of the object's motion is observed to reduce with a scale factor of 0.98 from one oscillation to the next.

(iii) Find the value of k. [3]

$$e^{-\frac{2\pi}{3}k} = 0.98$$

$$-\frac{2\pi}{3}k = \ln(0.98)$$

$$\Rightarrow k = -\frac{3}{2\pi}\ln(0.98) = \underline{\underline{0.009646}}$$

At the start of the object's motion, $x = 0$ and the velocity is 12 cms^{-1} in the positive x direction.

(iv) Find an equation for x as a function of t. [4]

$$x = e^{-kt}(A\cos 3t + B\sin 3t)$$

$$t = 0 \quad x = 0$$

$$\Rightarrow \underline{\underline{A = 0}}$$

$$x = Be^{-kt}\sin 3t$$

$$\frac{dx}{dt} = -kBe^{-kt}\sin 3t + 3Be^{-kt}\cos 3t$$

$$= e^{-kt}(3B\cos 3t - kB\sin 3t)$$

$$t = 0 \quad \frac{dx}{dt} = v = 12 \Rightarrow 12 = e^{0} \times 3B \Rightarrow \underline{\underline{B = 4}}$$

$$\Rightarrow x = 4e^{-0.009646t}\sin 3t$$

(v) Without doing any further calculations, explain why, according to this model, the greatest distance of the object from its starting point in the subsequent motion will be slightly less than 4 cm. [2]

When $k = 0$, there is no damping and the amplitude is 4

$k < 0$, so there is a tiny amount of damping. This means the amplitude will be slightly less than 4.

END OF QUESTION PAPER